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We develop a general approximation framework is developed for use in optimal control problems governed by nonlinear functional differential equations. Our approach entails only the use of linear semigroup approximation results while the nonlinearities are treated as perturbations of a linear system. Numerical results are presented for several simple nonlinear optimal control problem examples.

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APPROXIMATION OF NONLINEAR FUNCTIONAL
DIFFERENTIAL EQUATION CONTROL SYSTEMS[1]

by

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June, 1977

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Abstract: We develop a general approximation framework for use in optimal control problems governed by nonlinear functional differential equations. Our approach entails only the use of linear semigroup approximation results while the nonlinearities are treated as perturbations of a linear system. Numerical results are presented for several simple nonlinear optimal control problem examples.

Keywords: Functional differential equations, nonlinear control problems, semigroups, approximation techniques.

1. Introduction

The purpose of this paper is to extend and provide proofs for the theoretical results announced earlier in Refs. 1 and 2 and to demonstrate that the approximation method thus proposed is a reasonable one for certain classes of nonlinear functional differential equation (FDE) control problems by reporting some of our numerical experience. The idea (which is fundamental to our efforts here) of approximating infinite dimensional FDE systems by finite dimensional ordinary differential equation (ODE) systems is not a new one and the reader should consult Ref. 3 for a fairly complete summary of the previous literature on this idea.

In the presentation to follow we shall develop a theoretical framework for nonlinear control problems and then offer sample numerical results for several simple examples. In developing the theoretical foundations in sections 2 and 3, we shall demonstrate that, for a reasonably generous class of interesting nonlinear systems, approximation ideas involving only linear semigroup theory (developed previously in Refs. 3 and 4) are adequate for development of an abstract nonlinear approximation framework. These ideas are not at all technically complicated but rather rely only on quite standard techniques from known ODE and FDE theory plus frequent use of the Gronwall inequality.

Among other authors (Refs. 5-9) who have developed directly related theoretical results for the approximation of nonlinear FDE systems, only Repin does not make use of the machinery of nonlinear semigroup theory. But the assumptions in the Repin presentation lead to results that are not directly

applicable to control problems. Thus while there is some overlap between our results and those found in each of the references cited above, our treatment differs from each of those in some essential aspects and it is accurate to say that our results neither completely subsume nor are they subsumed by the results (individually or collectively) of these authors.

A feature of our presentation is the simplicity of the ideas and proofs involved, given the linear theory of Ref. 3. Indeed, one should note the similarity between the classical techniques for existence, uniqueness and continuous dependence results used in section 2 (one could, of course, instead use there standard - and perhaps more elegant - fixed point type arguments) and those employed in the approximation arguments of section 3. A disadvantage of our formulation in this paper is that the present assumptions allow one to include discrete delay terms - $x(t-r)$ terms as opposed to x_t terms - only in the linear part of the control system equations. We are currently investigating to what extent our ideas can be further extended to treat general nonlinear differential-difference equation control systems.

The notation employed in the sequel is quite standard. $L_p([a,b],X)$ will denote the normed space of X-valued L_p "functions" (we shall not distinguish between representatives and equivalence classes since the meaning will always be apparent to the reader) defined on $[a,b]$ and $L_p^\nu(a,b)$ (or $L_p(a,b)$ if $\nu=1$) will sometimes be used when $X = R^\nu$ where R^ν is Euclidean space. The class of functions f with $|f|^p$ locally integrable will be denoted by

L_p^{loc} and those with p continuous derivatives will be denoted by $C^{(p)}$. Throughout, the space $Z = R^n \times L_2^n(-r,0)$ will be used with the usual product topology, i.e. if $z = (\eta,\phi) \in Z$, $|z|^2 = |\eta|^2 + |\phi|^2$. Here, as elsewhere, the general symbol $|\cdot|$ is used for the norm when no confusion will result. That is, unless otherwise stated explicitly, $|x|$ denotes the norm of $x \in X$ where the norm is the one understood for the space X. The space of continuous functions on [a,b] into Z with the usual supremum norm will be denoted by $\mathscr{C}([a,b],Z)$ and $AC^n[a,b]$ will be the space of R^n-valued absolutely continuous functions on [a,b] (with the supremum norm, i.e. $\mathscr{C}$ topology, unless otherwise specified). The vector space of $n \times m$ matrices $\mathscr{L}(R^m,R^n)$ will be written as $\mathscr{L}_{n,m}$. Finally for a given measurable function $s \to x(s)$, the symbol x_t denotes the measurable function on [-r,0] given by $\theta \to x_t(\theta) = x(t+\theta)$, $-r \le \theta \le 0$. For vectors we shall not in general distinguish between the column form and its transpose when the usage makes clear our intended meaning.

2. Abstract Formulation of the FDE Control System

We shall consider in the following discussions the basic nonlinear control system

$$\begin{aligned} \dot{x}(t) &= L(x_t) + f(t,x(t),x_t,u(t)), \quad t \in [0,t_1], \\ x(0) &= \eta, \quad x_o = \phi , \end{aligned} \tag{1}$$

where t_1 is finite, $(\eta,\phi) \in Z$, and the linear part L is as given in Ref. 3. That is, there are matrices $A_i \in \mathscr{L}_{n,n}$ and $D \in L_2([-r,0], \mathscr{L}_{n,n})$, and numbers h_i, $0 = h_o < h_1 < \ldots < h_\nu = r$, such that

$$L(\phi) = \sum_{i=o}^{\nu} A_i \phi(-h_i) + \int_{-r}^{0} D(\theta)\phi(\theta)d\theta$$

for $\phi \in L_2^n(-r,0)$ (where, of course, the proper interpretation must be given to the point evaluations of ϕ if this "function" is indeed only an L_2 "function"). The nonlinearities of the system are contained in the mapping $f: R^1 \times R^n \times L_2^n(-r,0) \times R^m \to R^n$, about which we make several fundamental standing assumptions for the presentations in this paper:

(H1) The mapping $(t,y,\psi,v) \to f(t,y,\psi,v)$ is continuous on $R^1 \times R^n \times L_2^n(-r,0) \times R^m$.

(H2) For any bounded subset $\mathscr{D}$ of Z there exist $m_i = m_i(\mathscr{D})$, $m_i \in L_\infty^{loc}$, i=1,2, such that for $v \in R^m$, $t \in R^1$, and $(x,\phi),(y,\psi) \in \mathscr{D}$ one has
$|f(t,x,\phi,v)-f(t,y,\psi,v)| \leq \{m_1(t)+m_2(t)|v|\}\{|x-y|+|\phi-\psi|\}$.

(H3) There is a continuous mapping $t \to B(t) \in \mathscr{L}_{n,m}$ such that $f(t,0,0,v) = B(t)v$ for all $(t,v) \in R^1 \times R^m$. Further, there exist functions $\hat{m}_i \in L_\infty^{loc}$, $i=1,2$ such that
$|f(t,x,\phi,v)| \leq \{\hat{m}_1(t)+\hat{m}_2(t)|v|\}\{|x|+|\phi|\} + |B(t)||v|$
for all t,v and all (x,ϕ) with $|(x,\phi)|_Z$ sufficiently large.

(H4) There is a continuous function $g:R^1 \times R^n \times L_2^n(-r,0) \to R^1$ such that
$|f(t,x,\phi,v)-f(t,x,\phi,u)| \leq g(t,x,\phi)|v-u|$
for all $(t,x,\phi) \in R^1 \times R^n \times L_2^n(-r,0)$ and $v,u \in R^m$.

Remark 2.1. We observe that hypotheses (H2) and (H3) together yield that the following condition is satisfied by f:

(G) There exist $\tilde{m}_1,\tilde{m}_2$ in L_∞^{loc} such that $|f(t,y,\psi,v)| \leq \{\tilde{m}_1(t)+\tilde{m}_2(t)|v|\}\{|y|+|\psi|\} + |B(t)||v|$ for all (t,y,ψ,v) in $R^1 \times R^n \times L_2^n(-r,0) \times R^m$.

Let π_1,π_2 denote the coordinate projections of $Z = R^n \times L_2^n(-r,0)$ onto R^n and L_2 respectively. That is, $\pi_1(\eta,\phi) = \eta$, $\pi_2(\eta,\phi) = \phi$. Define a mapping $F:R^1 \times Z \times R^m \to Z$ by

$$F(t,z,v) \equiv (f(t,\pi_1 z,\pi_2 z,v),0).$$

Several properties of the mapping F will be used in the sequel and we list these here for later reference.

P_1: For any bounded subset $\mathscr{D}$ of Z there exist M_1,M_2 (depending on $\mathscr{D}$) in L_∞^{loc} such that

$$|F(t,z,v)-F(t,w,v)| \leq \{M_1(t)+M_2(t)|v|\}|z-w|$$

for all $z,w \in \mathscr{D}$ and $t \in R^1$, $v \in R^m$.

P_2: For any $z \in \mathscr{C}([0,t_1],Z)$ and $u \in L_2^m(0,t_1)$, the mapping $t \to |F(t,z(t),u(t))|$ is in $L_1(0,t_1)$.

Property P_1 follows directly from (H2) while P_2 is a consequence of (H2) and (H3).

We introduce next the solution semigroup $S(t):Z \to Z$ of the linear part of (1) as used extensively in the discussions in Refs. 3 and 4. Thus for, $(\eta,\phi) \in Z$, we define $S(t)(\eta,\phi) = (x(t),x_t)$ for $t \geq 0$, where x is the solution of (1) on $[0,\infty)$ with $f \equiv 0$. It is easily shown that $\{S(t)\}_{t\geq 0}$ is a C_o-semigroup with infinitesimal generator $\mathscr{A}$ defined on $\mathscr{D}(\mathscr{A}) = \{(\eta,\phi)|\phi$ is absolutely continuous with $\dot{\phi} \in L_2^n(-r,0)$ and $\phi(0) = \eta\}$ by $\mathscr{A}(\phi(0),\phi) = (L(\phi),\dot{\phi})$. Let M,β be constants such that $|S(t)| \leq M\exp(\beta t)$ for $t \geq 0$.

We shall be concerned with an abstract form of (1) given by the implicit equation

$$z(t) = S(t)z_o + \int_o^t S(t-\sigma)F(\sigma,z(\sigma),u(\sigma))d\sigma \ . \tag{2}$$

That this equation is in some sense equivalent to (1) is the focus of the remaining discussions in this section.

Lemma 2.1. Under the hypotheses (H1)-(H3), equation (2) defines for each $u \in L_2^m(0,t_1)$ and $z_o \in Z$, a unique function $t \to z(t)$ in $\mathscr{C}([0,t_1],Z)$.

Proof: For any two continuous solutions z and w of (2) on $[0,t_1]$ we have using P_1 (since $\{z(t)\},\{w(t)\}$ lie in a bounded subset of Z)

$$|z(t)-w(t)| \leq \int_0^t M \exp[\beta(t-\sigma)]\{M_1(\sigma)+M_2(\sigma)|u(\sigma)|\}|z(\sigma)-w(\sigma)|d\sigma$$

$$= \int_0^t \mathscr{M}(\sigma)|z(\sigma)-w(\sigma)|d\sigma$$

where $\mathscr{M} \in L_1(0,t_1)$. Uniqueness of solutions thus follows from an application of Gronwall's inequality.

The proof of existence also involves only quite standard arguments. Define for k=0,1,2,... the Picard iterates $\{z^k\}$ by

$$z^o(t) = S(t)z_o$$

$$z^k(t) = S(t)z_o + \int_0^t S(t-\sigma)F(\sigma,z^{k-1}(\sigma),u(\sigma))d\sigma \;, \tag{3}$$

for $t \in [0,t_1]$. Clearly the iterates are well-defined and an inductive argument employing the strong continuity of $\{S(t)\}$ and property P_2 in the usual manner yield easily that $z^k \in \mathscr{C}([0,t_1],Z)$ for k=0,1,2,... . Using (G) in the second equation of (3) one obtains

$$|z^k(t)| \leq M \exp(\beta t)\left\{|z_o| + \int_0^t (2[\tilde{m}_1(\sigma)+\tilde{m}_2(\sigma)|u(\sigma)|]|z^{k-1}(\sigma)|+|B(\sigma)||u(\sigma)|)d\sigma\right\}$$

$$\leq M \exp(\beta t)\left\{|z_o| + \gamma + \int_0^t \tilde{\mathscr{M}}(\sigma)|z^{k-1}(\sigma)|d\sigma\right\}$$

where γ and $\tilde{\mathscr{M}} \in L_2$ are independent of k. A simple inductive argument using this inequality can be made; one obtains

$$|z^k(t)| \leq P_o M \exp(\beta t_1)\exp[H(t_1)]$$

where $p_o \equiv \sup\{|S(t)z_o| + \gamma : t \in [0,t_1]\}$ and $H(t) = M\exp(\beta t_1)\int_0^t \tilde{\mathscr{M}}(\sigma)d\sigma$.

Thus $\{z^k\}$ is bounded in $\mathscr{C}([0,t_1],Z)$.

Use of this boundedness along with property P_1 allows us to choose M_1, M_2 in $L_\infty(0,t_1)$ such that

$$|F(\sigma,z^k(\sigma),u(\sigma))-F(\sigma,z^{k-1}(\sigma),u(\sigma))|$$

$$\leq \{M_1(\sigma)+M_2(\sigma)|u(\sigma)|\}|z^k(\sigma)-z^{k-1}(\sigma)|$$

for all $k=1,2,\ldots$. This can be used to show that $\{z^k\}$ is in fact a Cauchy sequence in $\mathscr{C}([0,t_1],Z)$. Indeed, using completely straightforward arguments one can obtain

$$|z^k-z^{k-1}|_{\mathscr{C}} \leq \left\{M\exp(\beta t_1)\right\}^{k+1} q_o \frac{\mu(t_1)^k}{k!}$$

where $q_o \equiv \int_0^{t_1} |F(\sigma,z^o(\sigma),u(\sigma))|d\sigma$ and $\mu(t) \equiv \int_0^t \{M_1(\sigma)+M_2(\sigma)|u(\sigma)|\}d\sigma$.

By passing to the limit in (3) we find that the limit z of the Cauchy sequence $\{z^k\}$ is the desired solution of (2).

Returning to system (1), we observe that for a given $u \in L_2^m(0,t_1)$ and $\phi \in AC^n(-r,0)$, $\phi(0) = \eta$, standard arguments can be employed to establish (under hypothesis (H1)-(H4)) existence of a unique solution on finite intervals $[0,t_1]$. That such solutions are continuous in (ϕ,u) can also be shown.

Lemma 2.2. Consider the mapping $(\phi,u) \to (x(t;\phi,u),x_t(\phi,u))$ from $AC^n(-r,0)\times L_2^m(0,t_1) \to Z$ where x is the solution of (1) corresponding to (ϕ,u) under hypothesis (H1)-(H4). This mapping is continuous.

Proof: Fixing (ϕ,u) in $AC^n(-r,0)\times L_2^m(0,t_1)$ and denoting by $x=x(\phi,u)$ the corresponding solution of (1), we shall consider arguments involving $y=y(\psi,v)$, solutions of (1) corresponding to (ψ,v) in some bounded neighborhood $B_1\times B_2$ of (ϕ,u) in $AC^n(-r,0)\times L_2^m(0,t_1)$. Using the linearity of L, the growth condition (G), and Gronwall's inequality along with quite standard reasoning, it is easy to show that $\{y(t;\psi,v)\,|\,t\in[0,t_1],\psi\in B_1,v\in B_2\}$ lies in a bounded subset of R^n. That is, $\{(y(t;\psi,v),y_t(\psi,v))\,|\,t\in[0,t_1],\psi\in B_1,v\in B_2\}$ lies in a bounded subset of Z so that the local Lipschitz conditions of (H2) can be employed with the functions m_i independent of the choice (ψ,v) in $B_1\times B_2$.

From (1) we obtain immediately for $x=x(\phi,u),y=y(\psi,v)$

$$
\begin{aligned}
|x(t)-y(t)| &\leq |\phi(0)-\psi(0)| + \left|\int_0^t L(x_\sigma-y_\sigma)d\sigma\right| \\
&\quad + \left|\int_0^t [f(\sigma,x(\sigma),x_\sigma,u(\sigma))-f(\sigma,y(\sigma),y_\sigma,v(\sigma))]d\sigma\right| \\
&= T_1+T_2+T_3. \qquad (4)
\end{aligned}
$$

Using (H2) and (H4), we find

$$
\begin{aligned}
T_3 &\leq \\
&\int_0^t \Big\{|f(\sigma,x(\sigma),x_\sigma,u(\sigma))-f(\sigma,x(\sigma),x_\sigma,v(\sigma))| \\
&\qquad + |f(\sigma,x(\sigma),x_\sigma,v(\sigma))-f(\sigma,y(\sigma),y_\sigma,v(\sigma))|\Big\}d\sigma \\
&\leq \int_0^t \Big\{g(\sigma,x(\sigma),x_\sigma)\,|u(\sigma)-v(\sigma)|+2[m_1(\sigma)+m_2(\sigma)\,|v(\sigma)|]\,\big|(x(\sigma),x_\sigma)-(y(\sigma),y_\sigma)\big|_Z\Big\}d\sigma \\
&\leq |g(x)|_{L_2(0,t_1)}|u-v|_{L_2(0,t_1)} + \int_0^t \mathscr{K}_v(\sigma)\,\big|(x(\sigma),x_\sigma)-(y(\sigma),y_\sigma)\big|d\sigma .
\end{aligned}
$$

From the form of L we obtain

$$T_2 \le \int_0^t \left| \sum_{i=0}^{\nu} A_i(x(\sigma-h_i)-y(\sigma-h_i)) + \int_{-r}^{0} D(\theta)[x(\sigma+\theta)-y(\sigma+\theta)]d\theta \right| d\sigma$$

$$\le \int_0^t \{ \sum_0^{\nu} |A_i||x(\sigma)-y(\sigma)| + |D|_{L_2}|x_\sigma - y_\sigma|_{L_2}\} d\sigma$$

$$+ \int_{-r}^{0} \sum_{i=1}^{\nu} |A_i||x(\sigma)-y(\sigma)|d\sigma$$

$$\le r(\sum_1^{\nu}|A_i|)|x_0-y_0|_{\mathscr{C}} + \int_0^t E\left|(x(\sigma),x_\sigma)-(y(\sigma),y_\sigma)\right|_Z d\sigma$$

where $E \equiv 2\{\sum_0^{\nu} |A_i| + |D|\}$.

Using these estimates in (4) we thus have

$$|x(t)-y(t)| \le \Gamma|\phi-\psi|_{\mathscr{C}} + |g(x)||u-v|_{L_2} + \int_0^t (\mathscr{K}_v(\sigma)+E)\Delta(\sigma)d\sigma$$

where $\Delta(\sigma) \equiv |(x(\sigma),x_\sigma)-(y(\sigma),y_\sigma)|_Z$. But since $|x_t|_{L_2} \le \sqrt{r}|x_t|_{\mathscr{C}}$ this implies

$$\Delta(t) \le \sqrt{1+r}\left\{\Gamma|\phi-\psi| + |g(x)||u-v| + \int_0^t (\mathscr{K}_v(\sigma)+E)\Delta(\sigma)d\sigma\right\}.$$

With an application of Gronwall's inequality we then obtain

$$\Delta(t) \le \sqrt{1+r}\left\{\Gamma|\phi-\psi|+|g(x)||u-v|\right\} \exp\left\{\int_0^t (\mathscr{K}_v(\sigma)+E)d\sigma\right\},$$

from which the desired continuity follows easily once one notes that the exponential term is bounded uniformly in $v \in B_2$.

Arguments almost identical to those above (employing P_1 instead of (H2) along with Gronwall's inequality) can be used to establish

an analogous continuity result for the functions defined by (2). We shall therefore only state this continuity property, omitting the proof.

Lemma 2.3. The mapping $(\phi(0),\phi,u) \to z(t;\phi,u)$ where z is defined by (2) with $z_o = (\phi(0),\phi)$ is continuous on $\mathcal{D}(\mathcal{A}) \times L_2^m(0,t_1)$ in the $Z \times L_2$ topology (and hence in the $R^n \times \mathcal{C} \times L_2$ topology).

The preceding discussions allow us to establish with relative ease a desired equivalence between systems (1) and (2).

Theorem 2.1. Suppose f satisfies (H1)-(H4). Then for $(\eta,\phi) = (\phi(0),\phi) \in \mathcal{D}(\mathcal{A})$ and $u \in L_2^m(0,t_1)$ we have

$$z(t;\phi,u) = (x(t;\phi,u), x_t(\phi,u))$$

where $t \to x(t;\phi,u)$ is the solution of (1) and $t \to z(t;\phi,u)$ is defined by (2) with $z_o = (\phi(0),\phi)$.

Proof: We shall say that $t \to \zeta(t)$ is a <u>strong</u> solution (in Z) of

$$\begin{aligned} \dot{\zeta}(t) &= \mathcal{A}\zeta(t) + F(t,\zeta(t),u(t)) \\ \zeta(0) &= (\phi(0),\phi) \end{aligned} \tag{5}$$

if $t \to \zeta(t)$ is continuous for $t \geq 0$, $C^{(1)}$ for $t > 0$ with $\zeta(t) \in \mathcal{D}(\mathcal{A})$ for $t > 0$ and the equation in (5) is satisfied for $t > 0$.

We first consider solutions for $\phi \in C^{(1)}$ and $u \in C^{(0)}$ and define $t \to w(t)$ by $w(t) = (x(t;\phi,u), x_t(\phi,u))$ where x is the corresponding solution of (1). Direct computations for $\frac{d}{dt} w(t)$ in the Z-norm show that $\dot{w}(t) = \mathcal{A}w(t) + F(t,w(t),u(t))$ and that the necessary continuity requirements obtain so that $t \to w(t)$ is a strong solution

of (5). Next let $t \to \zeta(t)$ be any strong solution of (5) corresponding to $\phi \in C^{(1)}$, $u \in C^{(0)}$ and define $\lambda(\sigma) = S(t-\sigma)\zeta(\sigma)$ for $0 \leq \sigma \leq t$. We observe then that the assumptions on f (see (H1)) imply that in this case $\sigma \to F(\sigma,\zeta(\sigma),u(\sigma))$ is continuous. Again direct computations (e.g., for $\varepsilon>0$, write $\frac{1}{\varepsilon}[\lambda(\sigma+\varepsilon)-\lambda(\sigma)] = S(t-\sigma-\varepsilon)\{\frac{1}{\varepsilon}[\zeta(\sigma+\varepsilon)-\zeta(\sigma)] + \frac{1}{\varepsilon}[I-S(\varepsilon)]\zeta(\sigma)\}$ and take the limit as $\varepsilon \to 0^+$) reveal that the right derivative $\dot{\lambda}^+(\sigma)$ exists and equals the continuous function $\sigma \to S(t-\sigma)F(\sigma,\zeta(\sigma),u(\sigma))$. It follows in the usual manner that $\dot{\lambda}$ exists continuously and indeed for $t > 0$

$$\lambda(t)-\lambda(0) = \int_0^t \dot{\lambda}(\sigma)d\sigma = \int_0^t S(t-\sigma)F(\sigma,\zeta(\sigma),u(\sigma))d\sigma.$$

Since $\lambda(t) = S(0)\zeta(t), \lambda(0) = S(t)\zeta(0)$ we thus find that any strong solution ζ of (5) corresponding to $\phi \in C^{(1)}, u \in C^{(0)}$ must satisfy

$$\zeta(t) = S(t)\zeta(0) + \int_0^t S(t-\sigma)F(\sigma,\zeta(\sigma),u(\sigma))d\sigma .$$

By Lemma 2.1 we see that any such strong solution of (5) must be the unique solution of (2) in $\mathscr{C}([0,t_1],Z)$. But we saw above that $t \to w(t) = (x(t),x_t)$ is a strong solution of (5) for ϕ,u so restricted. It thus follows that $w(t) = (x(t),x_t) = z(t)$ where z is defined by (2) with $z_0 = (\phi(0),\phi)$ and the desired equivalence is obtained for $\phi \in C^{(1)}, u \in C^{(0)}$.

Finally, from Lemmas 2.2 and 2.3 we see that for each t the mappings $(\phi(0),\phi,u) \to (x(t),x_t), (\phi(0),\phi,u) \to z(t)$ from $\mathscr{D}(\mathscr{A}) \times L_2^m(0,t_1) \to Z$ are continuous in the $R^n \times \mathscr{C} \times L_2$ topology on $\mathscr{D}(\mathscr{A}) \times L_2$. This, combined with the denseness properties of the set

$\{(\phi(0),\phi,u) \mid \phi \in C^{(1)}, u \in C^{(0)}\}$ on which equivalence has already been proven, can be used to establish the equivalence as claimed in Theorem 2.1.

3. Approximation and Convergence

Having shown in the previous section that the control system

$$z(t) = S(t)z_o + \int_o^t S(t-\sigma)F(\sigma,z(\sigma),u(\sigma))d\sigma \tag{6}$$

is equivalent to our original system (1), we turn next to approximations of (1) via approximations of (6). Let $\{S^N(t)\}$, Z^N, P^N be a general approximating scheme as discussed in the first part of section 3 of Ref. 3. That is, $\{S^N(t)\}$ is a family of approximating C_o-semigroups satisfying $|S^N(t)| \leq M \exp(\beta t)$ and the convergence criteria $S^N(t)z \to S(t)z$, $z \in Z$, uniformly in t on finite intervals. The subspaces $Z^N \subset Z$ contain all elements of the form $(\xi,0)$, $\xi \in R^n$, and the linear operators $P^N: Z \to Z^N$ satisfy the condition: $P^N z_o \to z_o$ for all initial data of interest in studying the control system (6) (or (1)).

An example of such an approximating scheme is the "averaging" approximation scheme discussed in some detail in Ref. 3 (see also Ref. 4). While our discussions in this section will deal with general approximating schemes, we shall use the particular "averaging" scheme of Ref. 3 to discuss numerical results for nonlinear control systems in a subsequent section of this paper.

Given a general approximating scheme $\{S^N(t)\}$, Z^N, P^N as described above, we consider the approximating system

$$z^N(t) = S^N(t)P^N z_o + \int_o^t S^N(t-\sigma)F(\sigma,z^N(\sigma),u(\sigma))d\sigma. \tag{7}$$

Using arguments exactly as in the proof of Lemma 2.1, one can easily verify that (7) defines, for each N, a unique function

$z^N \in \mathscr{C}([0,t_1],Z)$. We shall show that for a given $u \in L_2^m(0,t_1)$, the solutions of (7) actually converge to the solution of (6).

Lemma 3.1. Suppose $\mathscr{S}$ is a bounded subset of $L_2^m(0,t_1)$. The sequence $\{z^N\}$ defined by (7) is bounded in $\mathscr{C}([0,t_1],Z)$, uniformly in $u \in \mathscr{S}$.

Proof: We have immediately from the assumptions on S^N, P^N that

$$|z^N(t)| \le M\exp(\beta t)\delta + \int_0^t M\exp[\beta(t-\sigma)]|F(\sigma,z^N(\sigma),u(\sigma))|d\sigma$$

where δ is independent of N and u.
But the condition (G) on f implies the existence of $\tilde{M}_1, \tilde{M}_2$ in L_∞^{loc} such that

$$|F(t,z,v)| \le \{\tilde{M}_1(t)+\tilde{M}_2(t)|v|\}|z|+|B(t)||v|.$$

Thus we find

$$|z^N(t)| \le M\exp(\beta t)\left\{\delta + \int_0^t |B(\sigma)||u(\sigma)|d\sigma\right\}$$

$$+ \int_0^t M\exp[\beta(t-\sigma)]\{\tilde{M}_1(\sigma)+\tilde{M}_2(\sigma)|u(\sigma)|\}|z^N(\sigma)|d\sigma$$

$$\le M\exp(\beta t)\{\delta+|B|_{L_2}|u|_{L_2}\} + \int_0^t k_u(\sigma)|z^N(\sigma)|d\sigma,$$

which upon an application of Gronwall's inequality yields

$$|z^N(t)| \le M\exp(\beta t)\{\delta+|B||u|\}\exp\left\{\int_0^{t_1} k_u(\sigma)d\sigma\right\}.$$

Since $k_u(\sigma) = M\exp(\beta t_1)\{\tilde{M}_1(\sigma)+\tilde{M}_2(\sigma)|u(\sigma)|\}$, it is clear that for $u \in \mathscr{S}$, the right side of this last inequality is bounded uniformly in u and N.

Lemma 3.2. For $\mathscr{U}$ bounded, $\{z^N\}$ is Cauchy in $\mathscr{C}([0,t_1],Z)$, uniformly in $u \in \mathscr{U}$.

Proof: For $u \in \mathscr{U}$ and each positive integer j we denote by g^j the function

$$g^j(\sigma) \equiv f(\sigma,\pi_1 z^j(\sigma),\pi_2 z^j(\sigma),u(\sigma)).$$

Since $|g^j(\sigma)| \leq 2\{\tilde{m}_1(\sigma)+\tilde{m}_2(\sigma)|u(\sigma)|\}|z^j(\sigma)|+|B(\sigma)||u(\sigma)|$ (see condition (G)) with $\tilde{m}_i \in L_\infty^{loc}$ and $\{z^j\}$ uniformly bounded, it is easily argued that $g^j \in L_2^n(0,t_1)$ with

$$|g^j|_{L_2} \leq \Lambda$$

uniformly in $u \in \mathscr{U}$ and $j=1,2,\ldots$.

Let $T^N(t):R^n \to Z^N$ and $T(t):R^n \to Z$ be defined by $T^N(t)\xi \equiv S^N(t)(\xi,0)$, $T(t)\xi \equiv S(t)(\xi,0)$ where $\xi \in R^n$.

From Lemma 3.2 of Ref. 4, one has that for every t, $T^N(t) \to T(t)$ in the uniform operator norm. In fact, one actually has for each $t \in [0,t_1]$

$$\int_0^t |T^N(t-\sigma)-T^{N+K}(t-\sigma)|^2 d\sigma \leq \int_0^{t_1} |T^N(t_1-\sigma)-T^{N+K}(t_1-\sigma)|^2 d\sigma = \varepsilon_2(N,K)$$

where $\varepsilon_2(N,K) \to 0$ as $N,K \to \infty$.

Furthermore, the convergence properties of $\{S^N(t)\}$ and P^N imply for $z_o \in Z$ a fixed initial value and $t \in [0,t_1]$

$$|S^N(t)P^N z_o - S^{N+K}(t)P^{N+K}z_o| \leq \varepsilon_1(N,K)$$

where $\varepsilon_1(N,K) \to 0$ as $N,K \to \infty$.

Thus, for $N,K > 0$ and $t \in [0,t_1]$ we may write

$$|z^N(t)-z^{N+K}(t)| \leq |(S^N(t)P^N-S^{N+K}(t)P^{N+K})z_o|$$

$$+ \int_o^t |S^N(t-\sigma)(g^N(\sigma),0)-S^{N+K}(t-\sigma)(g^{N+K}(\sigma),0)|d\sigma$$

$$\leq \varepsilon_1(N,K) + \int_o^t |T^N(t-\sigma)g^N(\sigma)-T^{N+K}(t-\sigma)g^{N+K}(\sigma)|d\sigma$$

$$\leq \varepsilon_1(N,K) + \int_o^t |(T^N(t-\sigma)-T^{N+K}(t-\sigma))g^N(\sigma)|d\sigma$$

$$+ \int_o^t |T^{N+K}(t-\sigma)\{g^N(\sigma)-g^{N+K}(\sigma)\}|d\sigma$$

$$\leq \varepsilon_1(N,K) + \sqrt{\varepsilon_2(N,K)}\ |g^N|_{L_2} + \int_o^t M\exp[\beta(t-\sigma)]\,|g^N(\sigma)-g^{N+K}(\sigma)|d\sigma$$

$$\leq \varepsilon_1 + \sqrt{\varepsilon_2}\ \Lambda + \int_o^t M\exp[\beta(t-\sigma)]\,|g^N(\sigma)-g^{N+K}(\sigma)|d\sigma, \tag{8}$$

where $\varepsilon(N,K) = \varepsilon_1 + \sqrt{\varepsilon_2}\ \Lambda \to 0$ as $N,K \to \infty$, uniformly in $u \in \mathscr{U}$. From the boundedness property of $\{z^N\}$ and property P_1, we have

$$|g^N(\sigma)-g^{N+K}(\sigma)|_{R^n} = |F(\sigma,z^N(\sigma),u(\sigma))-F(\sigma,z^{N+K}(\sigma),u(\sigma))|_Z$$

$$\leq \{M_1(\sigma)+M_2(\sigma)|u(\sigma)|\}|z^N(\sigma)-z^{N+K}(\sigma)|$$

where $M_i \in L_\infty^{loc}$. Letting $\mathscr{M}_u(\sigma) \equiv M\exp(\beta t_1)\{M_1(\sigma)+M_2(\sigma)|u(\sigma)|\}$, we are thus able to rewrite the inequality (8) as

$$|z^N(t)-z^{N+K}(t)| \leq \varepsilon(N,K) + \int_o^t \mathscr{M}_u(\sigma)|z^N(\sigma)-z^{N+K}(\sigma)|d\sigma.$$

Once again appealing to the Gronwall inequality, we conclude

$$|z^N(t)-z^{N+K}(t)| \leq \varepsilon(N,K)\ \exp\left\{\int_0^t \mathscr{M}_u(\sigma)d\sigma\right\}.$$

It follows immediately that $\{z^N\}$ is Cauchy in $\mathscr{C}([0,t_1],Z)$, uniformly in $u \in \mathscr{U}$.

Theorem 3.1. Let z, z^N be defined by (6), (7) and let $\mathscr{U} \subset L_2^m(0,t_1)$ be bounded. Then $z^N(t;u) \to z(t;u)$ as $N \to \infty$, uniformly in t on $[0,t_1]$ and in $u \in \mathscr{U}$.

Proof: From the previous lemma we know that there exists $z \in \mathscr{C}([0,t_1],Z)$ satisfying $z^N(t;u) \to z(t;u)$ uniformly in t and in $u \in \mathscr{U}$. We claim that this limit function is the unique solution in $\mathscr{C}([0,t_1],Z)$ of (6) guaranteed to exist by Lemma 2.1. This follows immediately upon passing to the limit in (7) once one has observed that $F(\sigma,z^N(\sigma),u(\sigma)) \to F(\sigma,z(\sigma),u(\sigma))$ and $S^N(\tau)$ converges strongly to $S(\tau)$ with both sequences being dominated.

To make practical use of the convergence results of Theorem 3.1 in optimization problems, one must argue a little more than is promised in this theorem. More precisely, one wishes to replace the control system (1) by an equivalent system (6) and then solve the optimization problem governed by (6) (see Refs. 3 and 4). To obtain an approximate solution, one solves the optimization problem subject to (7). This results in a sequence $\{\bar{u}^N\}$ of (hopefully) approximating optimal controls and one desires, of course, that $z^N(t;\bar{u}^N) \to z(t;\bar{u})$ where $\bar{u}$ is the solution to the optimization problem constrained by (6). The sequence $\{\bar{u}^N\}$ is often (e.g., see Ref. 4) bounded in $L_2^m(0,t_1)$ so that weak convergence of

$\{\bar{u}^N\}$ (or of a subsequence at least) to $\bar{u}$ can be argued. From the inequality

$$|z^N(t;u^K)-z(t;u)| \leq |z^N(t;u^K)-z(t;u^K)| + |z(t;u^K)-z(t;u)|$$

and the convergence results of Theorem 3.1, it clearly suffices for our purposes to establish that $z(t;u^K) \to z(t;u)$ whenever $u^K \rightharpoonup u$ in $L_2^m(0,t_1)$. However it is not obvious from the implicit defining equation (6) for z that the dependence of z on u is even linear or affine and thus it is not surprising that further assumptions on f are required to obtain the desired results. For nonlinear systems which are affine in the control terms we can, as might be expected, show that $u^K \rightharpoonup u$ implies $z(t;u^K) \to z(t;u)$.

Assumption 3.1. Suppose that $f:R^1\times R^n\times L_2^n(-r,0)\times R^m \to R^n$ has the form

$$f(t,y,\psi,v) = \mathcal{N}_1(t,y,\psi) + \{\mathcal{N}_2(t,y,\psi) + B(t)\}v$$

$$= \mathcal{N}_1(t,y,\psi) + \mathcal{G}(t,y,\psi)v$$

where B is continuous and $\mathcal{N}_1:R^1\times Z \to R^n$, $\mathcal{N}_2:R^1\times Z \to \mathcal{L}_{n,m}$ satisfy:

(A1) The mappings $(t,y,\psi) \to \mathcal{N}_i(t,y,\psi)$, $i=1,2$, are continuous.

(A2) For $\mathcal{D}$ a bounded subset of Z, there exist L_∞^{loc} functions k_1,k_2 (depending possibly on $\mathcal{D}$) such that

$$|\mathcal{N}_i(t,y,\psi)- \mathcal{N}_i(t,x,\phi)| \leq k_i(t)|(x,\phi)-(y,\psi)|_Z$$

for all $(x,\phi),(y,\psi)\in\mathcal{D}$, $t\in R^1$, $i=1,2$.

(A3) $\mathcal{N}_i(t,0,0) = 0$, $i=1,2$ and there exist functions $\hat{k}_1,\hat{k}_2$ in L_∞^{loc} such that for $t\in R^1$,

$$|\mathcal{N}_i(t,y,\psi)| \le \hat{k}_i(t)\,|(y,\psi)|_Z$$

for all $(y,\psi) \in Z$ with $|(y,\psi)|$ sufficiently large, $i=1,2$.

It is not difficult to verify that any f satisfying Assumption 3.1 will also satisfy hypotheses (H1)-(H4) of section 2.

Theorem 3.2. Suppose f satisfies Assumption 3.1 and z is defined for $u \in L_2^m(0,t_1)$ by (6). Then $u^K \rightharpoonup u$ in $L_2^m(0,t_1)$ implies $z(t;u^K) \to z(t;u)$ as $K \to \infty$.

Proof: We first observe that $\{u^K\}$ bounded in L_2 along with condition (G) and Gronwall's inequality (again!) imply that $\{z(t;u^K)\,|\,t \in [0,t_1],\ K=1,2,\ldots\}$ lies in a bounded subset of Z (see the proof of Lemma 2.2) so that we can employ the *local* Lipschitz conditions of (A2) in our arguments. Letting $\tilde{\mathcal{N}}_1(t,z) \equiv (\mathcal{N}_1(t,\pi_1 z,\pi_2 z),0)$, $\tilde{\mathcal{G}}(t,z)v \equiv (\mathcal{G}(t,\pi_1 z,\pi_2 z)v,0)$, $z_K(t) = z(t;u^K)$, and $z(t) = z(t;u)$, we have

$$|z_K(t)-z(t)|$$

$$= \left|\int_0^t S(t-\sigma)\left\{\tilde{\mathcal{N}}_1(\sigma,z_K(\sigma))-\tilde{\mathcal{N}}_1(\sigma,z(\sigma))+\tilde{\mathcal{G}}(\sigma,z_K(\sigma))u^K(\sigma)-\tilde{\mathcal{G}}(\sigma,z(\sigma))u(\sigma)\right\}d\sigma\right|$$

$$\le \int_0^t M\exp[\beta(t-\sigma)]k_1(\sigma)\,|z_K(\sigma)-z(\sigma)|\,d\sigma$$

$$+ \left|\int_0^t S(t-\sigma)\left\{[\tilde{\mathcal{G}}(\sigma,z_K(\sigma))-\tilde{\mathcal{G}}(\sigma,z(\sigma))]u^K(\sigma)+\tilde{\mathcal{G}}(\sigma,z(\sigma))\,[u^K(\sigma)-u(\sigma)]\right\}d\sigma\right|$$

$$\le \int_0^t M\exp(\beta t)\{k_1(\sigma)+k_2(\sigma)\,|u^K(\sigma)|\}\,|z_K(\sigma)-z(\sigma)|\,d\sigma$$

$$+ \left|\int_0^t S(t-\sigma)(\Delta_K(\sigma),0)\,d\sigma\right| ,$$

where $\Delta_K(\sigma) \equiv \{\mathcal{N}_2(\sigma,\pi_1 z(\sigma),\pi_2 z(\sigma)) + B(\sigma)\}\{u^K(\sigma)-u(\sigma)\}$.

But from Theorem 3.2 of Ref. 4 we find that the operator $\mathcal{S}: L_2^n(0,t_1) \to \mathcal{C}([0,t_1],Z)$ given by

$$\mathcal{S}(g)(t) = \int_0^t S(t-\sigma)(g(\sigma),0)d\sigma$$

is a compact linear operator; thus $\Delta_K \longrightarrow 0$ in $L_2^n(0,t_1)$ implies $\mathcal{S}(\Delta_K)(t) \to 0$ uniformly in t on $[0,t_1]$. It follows that

$$|z_K(t)-z(t)| \leq \varepsilon_K + \int_0^t M \exp(\beta t_1)\{k_1(\sigma)+k_2(\sigma)|u^K(\sigma)|\}|z_K(\sigma)-z(\sigma)|d\sigma$$

where $\varepsilon_K \to 0$ as $K \to \infty$, so that one final application of the Gronwall inequality yields

$$|z_K(t)-z(t)| \leq \varepsilon_K \exp\left\{\int_0^{t_1} \mathcal{M}_K(\sigma)d\sigma\right\}$$

and the right side of this inequality $\to 0$ as $K \to \infty$, uniformly in t on $[0,t_1]$.

4. Numerical Results

One specific choice of $\{S^N(t)\}$, Z^N, P^N in the approximation scheme described in section 3 leads to the so-called "averaging" approximations that we have discussed in some detail in previous efforts dealing with linear systems (see Refs. 3 and 4). We turn now to a summary of some of our numerical results for nonlinear systems obtained using these averaging approximations in connection with the general nonlinear approximation scheme developed above. We first recall briefly the specific class of approximations of interest to us here. We do this for the linear operator L representing a simple differential-difference equation (for a discussion of the general linear case, see Ref. 3) given by

$$L(\phi) = A_o\phi(0) + A_1\phi(-r). \tag{9}$$

For each positive integer N, let $\{t_j^N\}$ be the partition of $[-r,0]$ given by $t_j^N = -\frac{jr}{N}$, $j=0,1,2,\ldots,N$, and define χ_j^N to be the characteristic function of $[t_j^N,t_{j-1}^N)$ for $j=2,\ldots,N$ and that of $[t_1^N,t_0^N] = [-\frac{r}{N},0]$ for $j=1$. Let $Z^N = \{(\eta,\phi)\,|\,\eta\in R^n,$ $\phi = \sum_{j=1}^{N} v_j^N\chi_j^N,\ v_j^N\in R^n\}$ and $P^N(\eta,\phi) = (\eta, \sum_1^N \phi_j^N\chi_j^N)$ where

$$\phi_j^N = \frac{N}{r}\int_{t_j^N}^{t_{j-1}^N} \phi(s)ds.$$

Define the semigroups $S^N(t) = \exp(\mathscr{A}^N t)$ where $\mathscr{A}^N : Z \to Z$ is given by

$$\mathscr{A}^N(\eta,\phi) = \left(A_o\eta+A_1\phi_N^N,\ \sum_1^N \frac{N}{r}\{\phi_{j-1}^N-\phi_j^N\}\chi_j^N\right)$$

with $\phi_o^N \equiv \eta$. It is shown in Ref. 3 that these choices lead to a scheme satisfying the convergence requirements of section 3.

Consider next application of these ideas to systems (1) with L given by (9) and $\eta = \phi(0)$. That is,

$$\begin{aligned} \dot{x}(t) &= A_o x(t)+A_1 x(t-r)+f(t,x(t),x_t,u(t)) \\ x(0) &= \eta,\ x_o = \phi\ . \end{aligned} \tag{10}$$

Defining the functions $e_o^N = (1,0)$, $e_j^N = (0,\chi_j^N)$, $j=1,2,\ldots,N$, and letting $z^N(t)$ in (7) be represented by

$z^N(t) = \sum_{j=o}^{N} w_j^N(t)e_j^N$, where $w_j^N(t) \in R^n$, it is not difficult to argue that the approximating system (7) for (10) can be written as an ordinary differential equation for the "Fourier" coefficients w_j^N. Thus, the approximation to (10) is equivalent to (see Ref. 4)

$$\begin{aligned} \dot{w}_o^N(t) &= A_o w_o^N+A_1 w_N^N(t)+f(t,w_o^N(t), \sum_1^N w_j^N(t)\chi_j^N,u(t)) \\ \dot{w}_j^N(t) &= \frac{N}{r}\{w_{j-1}^N(t)-w_j^N(t)\},\quad j=1,2,\ldots,N, \\ w^N(0) &= col(\phi(0),\phi_1^N,\ldots,\phi_N^N). \end{aligned} \tag{11}$$

For f satisfying the hypotheses of section 2, $(\phi(0),\phi) \in \mathcal{D}(\mathcal{A})$ and $u \in \mathcal{U}$, the convergence guaranteed by Theorem 3.1 implies that $w_o^N(t) \to x(t)$ uniformly in t on $[0,t_1]$ and in $u \in \mathcal{U}$, where w_o^N, x are the solutions of (11) and (10) respectively. We first present results for an example which illustrates this convergence of w_o^N to x in a system not subject to optimization.

Example 4.1. Consider

$$\dot{x}(t) = -1.5x(t)-1.25x(t-1)+x(t)\sin x(t),\ t > 0$$

$$x(\theta) = 10\theta+1, \quad -1 \le \theta \le 0,$$

with corresponding approximating equations

$$\dot{w}_o^N(t) = -1.5w_o^N(t)-1.25w_N^N(t)+w_o^N(t)\sin w_o^N(t)$$

$$\dot{w}_j^N(t) = N\{w_{j-1}^N(t)-w_j^N(t)\}, \qquad j=1,2,\ldots,N.$$

In Table 1 we list selected values for the variables w_o^N corresponding to several values of N. These are compared to the "true" solution values x which were obtained using one of the block methods (of fourth order) developed by Tavernini (see Ref. 10, pp. 77-78) directly on the delay system.

Table 1. Results for Example 4.1.

Time	w_o^8	w_o^{16}	w_o^{20}	w_o^{32}	x
0.0	1.0000	1.0000	1.0000	1.0000	1.0000
0.5	3.1589	3.2363	3.2519	3.2754	3.2532
1.0	2.2586	2.2808	2.2871	2.2989	2.3856
1.5	0.6754	0.4929	0.4488	0.3755	0.3446
2.0	-0.6484	-0.8157	-0.8503	-0.9026	-0.9137
2.5	-0.6967	-0.7269	-0.7312	-0.7363	-0.7655
3.0	-0.2656	-0.1766	-0.1524	-0.1106	-0.1060
3.5	0.1109	0.2418	0.2737	0.3257	0.3558
4.0	0.2454	0.3286	0.3463	0.3735	0.4017
4.5	0.1626	0.1547	0.1492	0.1377	0.1426
5.0	0.0137	-0.0478	-0.0644	-0.0923	-0.1054

The above example typifies some of the worst convergence behavior we have observed in using the averaging approximations to integrate nonlinear equations. In our efforts at Brown and in numerical work in collaboration with F. Kappel and H. Fröschl of the University of Graz, we have applied these approximations to a number of other nonlinear FDE examples (2 and 3-dimensional vector equations and other types of nonlinearities typical of those arising in several areas of applications). In these instances we found convergence of the approximations no worse than that illustrated in Example 4.1 and in a majority of the cases convergence was actually much more rapid than in the example detailed above.

We have also used the "averaging" approximations to solve for optimal controls in problems governed by nonlinear delay-differential control systems. Typical examples of interest are of the form (for more general problems to which our framework in sections 2 and 3 is applicable, see Refs. 3 and 4):

$$\text{Minimize } J = \frac{1}{2}\, x(t_1)Gx(t_1) + \frac{1}{2}\int_0^{t_1} \{x(t)Qx(t)+u(t)Ru(t)\}dt$$

subject to (10) where $u \in \mathcal{U}$, $\mathcal{U}$ is a closed convex subset of $L_2^m(0,t_1)$. Here we assume $G,Q \geq 0$, $R > 0$.

The approximating problems (for the averaging approximations) are given by:

$$\text{Minimize } J^N = \frac{1}{2}\, w_o^N(t_1)Gw_o^N(t_1) + \frac{1}{2}\int_0^{t_1} \{w_o^N(t)Qw_o^N(t)+u(t)Ru(t)\}dt$$

subject to (11) where again $u \in \mathcal{U}$.

Let $\bar{u}^N$ be a solution of the approximating problem (while standard existence arguments are applicable here, the usual uniqueness assertions are not appropriate since J^N is not necessarily even convex in the control u) with index N. Denote by $\bar{x}^N$ ($=\bar{w}_o^N$) the corresponding trajectory and $\bar{J}^N = J^N(\bar{u}^N)$ the corresponding value of the cost functional. Then under Assumption 3.1 of section 3, Theorems 3.1 and 3.2 along with standard arguments (see Theorem 4.1 of Ref. 3) can be employed to conclude that a subsequence of $\{(\bar{u}^N,\bar{x}^N)\}$ converges to a solution (u^*,x^*) of the original problem governed by the system (10) (the sequence $\{(\bar{u}^N,\bar{x}^N)\}$ itself converges if the solution (u^*,x^*) of the original problem is unique). We present results for several examples to illustrate this use of the approximating scheme in such optimization problems.

Example 4.2. Take as the original problem the task of minimizing

$$J = \frac{1}{2}\,x(2)^2 + \frac{1}{2}\int_0^2 u(t)^2dt$$

over $u \in \mathscr{U} = L_2(0,2)$ subject to

$$\begin{aligned} \dot{x}(t) &= \sin x(t)+x(t-1)+u(t), && 0 \le t \le 2, \\ x(\theta) &= 1, && -1 \le \theta \le 0. \end{aligned} \tag{12}$$

The corresponding approximating problem entails minimization of

$$J^N = \frac{1}{2}[w_o^N(2)]^2 + \frac{1}{2}\int_0^2 u(t)^2dt$$

over the same class of controls but subject to

$$\dot{w}_o^N(t) = \sin w_o^N(t) + w_N^N(t) + u(t)$$

$$\dot{w}_j^N(t) = N\{w_{j-1}^N(t) - w_j^N(t)\} \qquad j=1,2,\ldots,N, \tag{13}$$

$$w^N(0) = \mathrm{col}(1,1,\ldots,1).$$

One can apply the necessary conditions for nonlinear differential equation control problems along with standard computational routines (gradient, conjugate-gradient) - see Ref. 11 and Chapter X of Ref. 12 - to obtain extremals $(\bar{u}^N, \bar{w}^N)$ for the approximating problems This was done (in a manner described in more detail below) to obtain the values reported in the tables presented in this paper.

One can, on the other hand, use the necessary conditions for optimality of delay system problems (see Ref. 13, Theorem VII.2.31) and work directly with the original optimization problem. If (u^*,x^*) is a solution of this problem, these necessary conditions guarantee the existence of a multiplier λ such that u^* maximizes $H = -\frac{1}{2}u^2 + \lambda\{\sin x^* + x^*(t-1) + u\}$ (i.e. $u^*=\lambda$) where λ satisfies

$$\dot{\lambda}(t) = -\lambda(t)\cos x^*(t) - \lambda(t+1), \quad 0 \le t \le 2,$$

$$\lambda(t) = 0, \quad t > 2, \tag{14}$$

$$\lambda(2) = -x^*(2),$$

with x^* the solution of (12) corresponding to $u=u^*=\lambda$. Thus, we see that direct solution of the minimization problem for extremal pairs (u^*,x^*) requires solution of the mixed advanced-delayed system two-point boundary value problem (TPBVP) consisting of (12)

(with $u=\lambda$) and (14). In the results listed below (for this and other examples), we have given values for an <u>approximate</u> numerical solution of the associated TPBVP under the columns headed x^* and λ^*. These values can be used as a rough but independent check on the convergence of the trajectories $\bar{w}_o^N$ and controls $\bar{u}^N$ where $(\bar{u}^N, \bar{w}^N)$ are the computed extremal pairs for the approximating problems. In Table 2 we give a summary of selected values for the controls and in Table 3 we give corresponding trajectory values along with the optimal payoff values $\bar{J}^N$. For this example we see that the approximate solution for N=4 is a reasonably good one.

Table 2. Results for Example 4.2.

Time	$\bar{u}^4$	$\bar{u}^8$	$\bar{u}^{20}$	$\lambda^* \approx u^*$
0.00	-3.5737	-3.5704	-3.5663	-3.5686
0.25	-2.7615	-2.7706	-2.7769	-2.7779
0.50	-2.0473	-2.0594	-2.0664	-2.0561
0.75	-1.4938	-1.5052	-1.5073	-1.4849
1.00	-1.0889	-1.1018	-1.1041	-1.0577
1.25	-0.8030	-0.8226	-0.8371	-0.8197
1.50	-0.6067	-0.6337	-0.6558	-0.6400
1.75	-0.4722	-0.4994	-0.5180	-0.5010
2.00	-0.3751	-0.3981	-0.4112	-0.3938

Table 3. Results for Example 4.2.

Time	$\bar{w}_o^4$	$\bar{w}_o^8$	$\bar{w}_o^{20}$	x^*
0.00	1.0000	1.0000	1.0000	1.0000
0.25	0.6381	0.6372	0.6367	0.6270
0.50	0.4098	0.4081	0.4059	0.3880
0.75	0.2882	0.2932	0.2948	0.2743
1.00	0.2396	0.2581	0.2749	0.2747
1.25	0.2360	0.2661	0.2961	0.3172
1.50	0.2599	0.2932	0.3231	0.3415
1.75	0.3043	0.3336	0.3558	0.3672
2.00	0.3709	0.3939	0.4099	0.4187
$\bar{J}^N$	2.8759	2.9176	2.9437	

Example 4.3. In this example and the next, we choose a system with nonlinearity satisfying a local but not global Lipschitz condition. In the case where the approximation to the initial data is good, the approximation method is quite good for low values of N. The problem is to minimize

$$J = \frac{1}{2}\, x(2)^2 + \frac{1}{2}\int_0^2 \{x(t)^2 + u(t)^2\}dt$$

over $u \in \mathcal{U} = L_2(0,2)$ subject to

$$\begin{aligned} \dot{x}(t) &= x(t)\sin x(t) + x(t-1) + u(t), && 0 \le t \le 2, \\ x(\theta) &= 10, && -1 \le \theta \le 0. \end{aligned} \tag{15}$$

The approximating problem is one of minimizing

$$J^N = \frac{1}{2}[w_o^N(2)]^2 + \frac{1}{2}\int_0^2 \{w_o^N(t)^2 + u(t)^2\}dt$$

over $\mathcal{U}$ but subject to

$$\begin{aligned} \dot{w}_o^N(t) &= w_o^N(t)\sin w_o^N(t) + w_N^N(t) + u(t) \\ \dot{w}_j^N(t) &= N\{w_{j-1}^N(t) - w_j^N(t)\} \qquad j=1,2,\ldots,N, \\ w^N(0) &= \mathrm{col}(10,\ldots,10). \end{aligned} \tag{16}$$

In Table 4 we give values for $\bar{w}_o^8$, $\bar{u}^8$ and x^*,λ^* only since this suffices to demonstrate the approximation scheme as applied to this example. We also made computations for N = 16 and 32 and obtained values for $\bar{w}_o^{16},\bar{w}_o^{32},\bar{u}^{16},\bar{u}^{32}$ that differed only slightly from $\bar{w}_o^8,\bar{u}^8$ and x^*,u^*. Values for the payoff were

$\bar{J}^8 = 162.019$, $\bar{J}^{16} = 162.018$, $\bar{J}^{32} = 162.015$. Graphs of $\bar{u}^8$ and u* reveal strikingly good qualitative agreement between these two functions (see Figure 1).

Table 4. Results for Example 4.3.

Time	$\bar{u}^8$	$\lambda^* \approx u^*$	$\bar{w}_o^8$	x^*
0.00	-1.4959	-1.5159	10.0000	10.0000
0.25	-1.7215	-1.7275	10.3038	10.2700
0.50	-1.7683	-1.8690	10.3399	10.3214
0.75	-1.7787	-2.0592	10.3470	10.3164
1.00	-1.7465	-1.6550	10.3589	10.3184
1.25	-1.6812	-1.6677	10.3790	10.3750
1.50	-1.7365	-2.0244	10.3927	10.3770
1.75	-2.5597	-3.3634	10.3485	10.2916
2.00	-9.8694	-9.9181	9.9181	9.9174

Example 4.4. This example is exactly the same as that in Example 4.3 except for a change in initial data. In (15) we choose initial function $x_o = \phi$ where

$$\phi(\theta) = \begin{cases} 10(\theta+1) & -1 \leq \theta \leq -.5 \\ -10\theta & -.5 \leq \theta \leq 0 \end{cases}$$

with a corresponding change in (16) given by

$$w^N(0) = \mathrm{col}(\phi(0),\phi_1^N,\ldots,\phi_N^N).$$

Our numerical results are presented in Tables 5 and 6. The approximate numerical solution λ^*, x^* to the TPBVP for the original problem is not a particularly good one (one should have $\lambda^*(2) = -x^*(2)$ and we have a relative error here of about 5%) but it is adequate for an independent check that the method is producing solutions converging to an extremal for the original problem.

The averaging method does not approximate this initial data particularly well for low values of N and as one might expect, somewhat higher values of N are required to obtain an approximate solution of accuracy comparable to that in the previous example.

Table 5. Results for Example 4.4.

Time	$\bar{u}^8$	$\bar{u}^{16}$	$\bar{u}^{32}$	$\bar{u}^{48}$	$\lambda^* \approx u^*$
0.00	-2.3083	-2.3073	-2.3032	-2.3015	-2.3321
0.25	-2.1804	-2.2375	-2.2706	-2.2833	-2.3238
0.50	-1.9855	-2.1114	-2.1972	-2.2316	-2.2768
0.75	-1.5751	-1.6470	-1.6908	-1.7064	-1.7374
1.00	-1.1446	-1.1417	-1.1318	-1.1252	-1.1245
1.25	-0.8173	-0.7891	-0.7697	-0.7622	-0.7702
1.50	-0.5967	-0.5761	-0.5667	-0.5642	-0.5691
1.75	-0.4401	-0.4321	-0.4327	-0.4347	-0.4374
2.00	-0.3125	-0.3098	-0.3147	-0.3162	-0.3144

Table 6. Results for Example 4.4.

Time	$\bar{w}_o^8$	$\bar{w}_o^{16}$	$\bar{w}_o^{32}$	$\bar{w}_o^{48}$	x^*
0.00	0.0000	0.0000	0.0000	0.0000	0.0000
0.25	-0.1062	-0.1684	-0.2056	-0.2188	-0.2506
0.50	0.1351	0.1308	0.1245	0.1209	0.1086
0.75	0.3456	0.4290	0.4917	0.5167	0.5831
1.00	0.4120	0.4976	0.5550	0.5767	0.6406
1.25	0.3805	0.4130	0.4232	0.4244	0.4286
1.50	0.3269	0.3161	0.2966	0.2869	0.2586
1.75	0.2988	0.2803	0.2644	0.2578	0.2421
2.00	0.3106	0.3078	0.3115	0.3144	0.3313
$\bar{J}^N$	2.1765	2.3105	2.4012	2.4371	

All computations for the numerical results presented in this paper were carried out on the IBM 360/67 at Brown University. We are grateful to Mr. Douglas Reber for his assistance in making these computations. Indeed Mr. Reber developed with care and patience the software packages that were used for the examples discussed above as well as for some preliminary numerical studies (in our joint efforts with F. Kappel) of other examples involving use of several spline type approximations in an abstract framework similar to that discussed in sections 2 and 3 above and Ref. 4. We give now a brief description of the computational procedures followed in producing the results detailed above.

We employed an iterative optimization scheme that combined gradient (G) and conjugate-gradient (CG) steps in computing the extremals $(\bar{u}^N, \bar{w}^N)$ for the approximating ODE problems. For the results reported above, the first, fourth, seventh, etc. iterative steps were G steps while the remaining steps were CG steps formulated (a continuous problem version) according to the scheme given in Chapter X of Ref. 12. Integration of all ODE's within these steps was performed by a standard modification (Gill's) of a fourth-order Runge-Kutta method.

Iteration was continued until a convergence criteria (for changes in J^N) was satisfied. In the examples above this resulted in at most 10 iterative steps in each computation of a $(\bar{u}^N, \bar{w}^N)$ for Example 4.2, at most 5 steps in those of Example 4.3 and at most 13 steps in Example 4.4. CPU time required for these computations in the optimization package ran from 82 seconds for $(\bar{u}^4, \bar{w}^4)$ of Example 4.2 to a maximum of 578 seconds for $(\bar{u}^{48}, \bar{w}^{48})$

in Example 4.4. As one might expect, higher dimensional examples require increasingly higher amounts of CPU time.

Integration of the λ and x equations in the TPBVP's for the original problems (e.g., see (14)) was carried out via the same block method used in integration of the delay system in Example 4.1.

Our experience with examples suggests that when employed with a reasonably efficient optimization scheme, the averaging approximations for nonlinear control problems enjoy about the same computational behavior as that already demonstrated by extensive studies on use of this method for linear control problems (see Refs. 3 and 14).

5. Concluding Remarks

The efforts involved in implementation and the convergence results obtained in practice with a number of examples lead us to believe that the approximation ideas of sections 2 and 3 are indeed reasonable possibilities when one is optimizing delay system problems. The framework developed here is an effective means of reducing the difficulties associated with certain nonlinear FDE control problems to those for nonlinear ODE control problems. While there are still a number of difficulties associated with these latter problems, a great deal more in the way of specific techniques is available in the literature for these problems than in the case of FDE problems.

The assumptions underlying development of the ideas in section 2 are not so restrictive as to preclude treatment of nonlinear systems of a type arising frequently in applications. Assumption (H2) is a local Lipschitz condition in $(x(t),x_t)$ in the Z-norm while (H4) is a growth condition not unlike those often imposed in ODE theory when one wishes to guarantee existence of solutions on any fixed finite interval. Among nonlinearities that fall within the restrictions of (H1)-(H4) are those (see Ref. 2) arising in enzyme kinetics, certain nonlinear protein synthesis models, particle accelerator models and bilinear control systems. Also included are nonlinearities involving terms of the form

$$f(t,y(t),y_t) = \frac{\mu y_1(t) \int_{-r}^{0} y_2(t+\theta)\gamma(\theta)d\theta}{K + \int_{-r}^{0} y_2(t+\theta)\gamma(\theta)d\theta}$$

which have been employed in vector system models ($y=(y_1,y_2)$) for low growth chemostats (Ref. 15). We are in fact currently using some of the approximation ideas from this paper to study the qualitative properties of such chemostat models.

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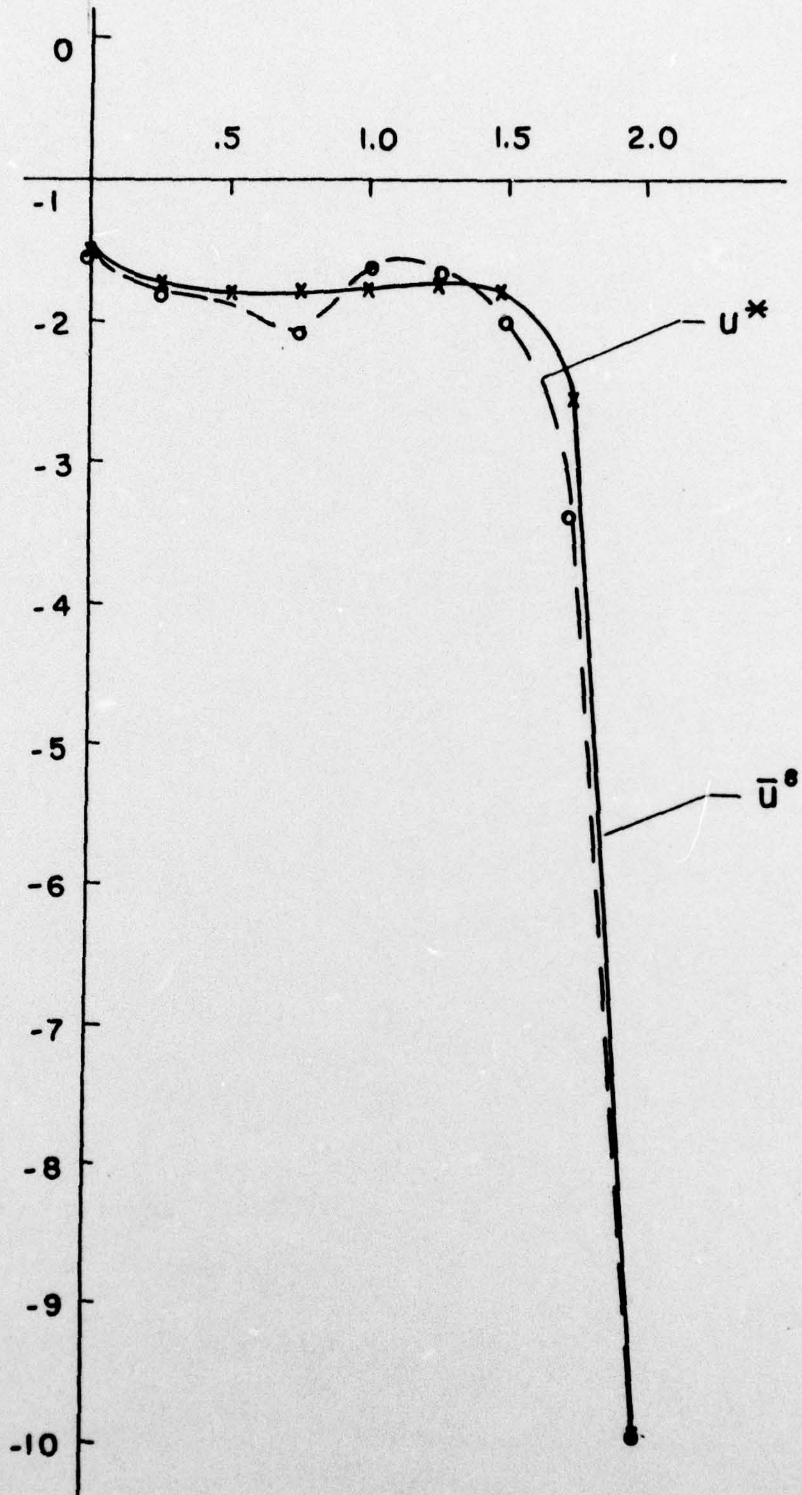

FIGURE 1. EXAMPLE 4.3

List of Captions for Figures

Figure 1. Controls for Example 4.3